Fonctions préharmoniques et applications conformes

Kilian Raschel

Journée en hommage à Jacques Neveu
Institut Henri Poincaré
23 mai 2017
Introduction & motivations

Applications in probability theory

Applications in combinatorics

Discrete harmonic functions in the quadrant
Introductory example & definition

Markov chains: example

1/2 1/2 1/2 1/2
1 2 3 4

1 & 4: absorbing states

Markov chains: general theorem

The hitting probabilities are characterized as being the minimal non-negative solutions to a system of linear recurrences.

Definition: $f$ harmonic if $L[f](x) = 0$ for all $x$ in a region $\subset \mathbb{Z}^d$.

$L[f](x) = \sum_{y \in N_y} p(y) \{f(x+y) - f(x)\}$,

with sets of neighbors $N_y \subset \mathbb{Z}^d$ and weights $p = \{p(y)\}_{y \in \mathbb{Z}^d}$.

Multivariate linear recurrences with constant coefficients

Bousquet-Mélou & Petkovšek '00
Introductory example & definition

Markov chains: example

\[ f_i = \mathbb{P}_i[\text{hit 4}] \text{ satisfies} \]
\[
\begin{align*}
  f_1 &= 0 \\
  f_4 &= 1 \\
  f_2 &= \frac{1}{2} f_1 + \frac{1}{2} f_3 \\
  f_3 &= \frac{1}{2} f_2 + \frac{1}{2} f_4
\end{align*}
\]

Solution: \( f_1 = 0, f_2 = \frac{1}{3}, f_3 = \frac{2}{3}, f_4 = 1 \)

1 & 4: absorbing states
**Introductory example & definition**

**Markov chains: example**

The hitting probabilities are characterized as being the **minimal non-negative solutions** to a system of **linear recurrences**.
Markov chains: example

Introductory example & definition

Markov chains: general theorem

The hitting probabilities are characterized as being the minimal non-negative solutions to a system of linear recurrences.

Definition: $f$ harmonic if $L[f](x) = 0$ for all $x$ in a region $\subset \mathbb{Z}^d$

\[ L[f](x) = \sum_{y \in N_y} p(y)\{ f(x+y) - f(x) \}, \]

with sets of neighbors $N_y \subset \mathbb{Z}^d$ and weights $p = \{ p(y) \}_{y \in \mathbb{Z}^d}$
**Introductory example & definition**

**Markov chains: example**

1/2 1/2 1/2

1 2 3 4

1/2

1 & 4: absorbing states

\[ f_i = \mathbb{P}_i[\text{hit } 4] \text{ satisfies } \]

\[
\begin{align*}
  f_1 &= 0 \\
  f_2 &= \frac{1}{2} f_1 + \frac{1}{2} f_3 \\
  f_3 &= \frac{1}{2} f_2 + \frac{1}{2} f_4 \\
  f_4 &= 1
\end{align*}
\]

Solution: \( f_1 = 0, f_2 = \frac{1}{3}, f_3 = \frac{2}{3}, f_4 = 1 \)

**Markov chains: general theorem**

The hitting probabilities are characterized as being the *minimal non-negative solutions* to a system of *linear recurrences*.

**Definition:** \( f \) harmonic if \( L[f](x) = 0 \) for all \( x \) in a region \( \subset \mathbb{Z}^d \)

\[
L[f](x) = \sum_{y \in N_y} p(y) \{ f(x + y) - f(x) \},
\]

with *sets of neighbors* \( N_y \subset \mathbb{Z}^d \) and *weights* \( p = \{ p(y) \}_{y \in \mathbb{Z}^d} \)

▷ Multivariate linear recurrences with constant coefficients

[ Bousquet-Mélou & Petkovšek '00 ]
Classical (continuous) harmonic functions in $\mathbb{R}^d$

$$\Delta[f](x) = \sum_{i=1}^{d} \frac{\partial^2 f(x)}{\partial x_i^2} = 0$$

- Possibility of adding weights $\sim$ elliptic operators
- Harmonic functions satisfy various properties: maximum principle/mean value property/Harnack inequalities/Liouville’s theorem/relations with analytic functions/etc.
- Examples of application: Heat equation/Dirichlet problem/Poisson’s equation/more general PDEs/etc.
History of Questions on preharmonic functions (1/2)

Classical (continuous) harmonic functions in $\mathbb{R}^d$

$$\Delta[f](x) = \sum_{i=1}^{d} \frac{\partial^2 f(x)}{\partial x_i^2} = 0$$

- Possibility of adding weights $\sim$ elliptic operators
- Harmonic functions satisfy various properties: maximum principle/mean value property/Harnack inequalities/Liouville’s theorem/relations with analytic functions/etc.
- Examples of application: Heat equation/Dirichlet problem/Poisson’s equation/more general PDEs/etc.

Do preharmonic functions satisfy similar properties?

- Dirichlet problem $\Diamond$ Phillips & Wiener ’23; Bouligand ’25
- Harnack inequalities $\Diamond$ Lawler & Polaski ’92; Varopoulos ’99
- Maximum principle, Liouville’s theorem & related topics $\Diamond$ Heilbronn ’48
- Cauchy-Riemann equations $\Diamond$ Duffin ’55; Kiselman ’05–’08
Further properties

- Rate of growth: Murdoch '63–'65; Ignatiuk-Robert '10
- Picard’s theorem (sign of harmonic functions) & factorization: Murdoch '63–'65
- Absolute monotonicity: Lippner & Mangoubi '15
### History of Questions on Preharmonic Functions (2/2)

#### Further Properties

- **Rate of growth**
  - Murdoch '63–'65; Ignatiuk-Robert '10

- **Picard’s theorem (sign of harmonic functions) & factorization**
  - Murdoch '63–'65

- **Absolute monotonicity**
  - Lippner & Mangoubi '15

#### Preharmonic & Harmonic Functions

- **Relations between discrete & continuous harmonic functions**
  - Lusternik '26; Ferrand '44; Kesten '91; Varopoulos '09
History of/Questions on preharmonic functions (2/2)

Further properties

- Rate of growth
  Murdoch '63–'65; Ignatiuk-Robert '10

- Picard’s theorem (sign of harmonic functions) & factorization
  Murdoch '63–'65

- Absolute monotonicity
  Lippner & Mangoubi '15

Preharmonic & harmonic functions

- Relations between discrete & continuous harmonic functions
  Lusternik '26; Ferrand '44; Kesten '91; Varopoulos '09

Probability theory models

- Ising models
  Mercat '01; Smirnov '10; Chelkak '11; Beffara '12

- Conformal invariance of lattice models
  Duminil-Copin & Smirnov '12
Further properties

- Rate of growth
  - Murdoch '63–'65; Ignatiuk-Robert '10

- Picard’s theorem (sign of harmonic functions) & factorization
  - Murdoch '63–'65

- Absolute monotonicity
  - Lippner & Mangoubi '15

Preharmonic & harmonic functions

- Relations between discrete & continuous harmonic functions
  - Lusternik '26; Ferrand '44; Kesten '91; Varopoulos '09

Probability theory models

- Ising models
  - Mercat '01; Smirnov '10; Chelkak '11; Beffara '12

- Conformal invariance of lattice models
  - Duminil-Copin & Smirnov '12

Special discrete functions

- Conformal mappings
  - Ferrand '44; Isaacs '52

- Discrete harmonic polynomials & discrete exponential functions
  - Terracini '45–'46; Heilbronn '48; Isaacs '52; Duffin '55; Duffin & Peterson '68
Preharmonic functions (2/2)

**Further properties**

- Rate of growth  
  Murdoch '63–'65; Ignatiuk-Robert '10

- Picard’s theorem (sign of harmonic functions) & factorization  
  Murdoch '63–'65

- Absolute monotonicity  
  Lippner & Mangoubi '15

**Preharmonic & harmonic functions**

- Relations between discrete & continuous harmonic functions  
  Lusternik '26; Ferrand '44; Kesten '91; Varopoulos '09

**Probability theory models**

- Ising models  
  Mercat '01; Smirnov '10; Chelkak '11; Beffara '12

- Conformal invariance of lattice models  
  Duminil-Copin & Smirnov '12

**Special discrete functions**

- Conformal mappings  
  Ferrand '44; Isaacs '52

- Discrete harmonic polynomials & discrete exponential functions  
  Terracini '45–'46; Heilbronn '48; Isaacs '52; Duffin '55; Duffin & Peterson '68

**Potential theory**

- Martin boundary  
  Woess '92; Kurkova & Malyshev '98; Ignatiuk-Robert & Loree '10; Mustapha '15
Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases

- \( q(n; i, j) = \#_{\mathbb{N}^2} \{(0, 0) \xrightarrow{n} (i, j)\} \)
- \( q(n + 1; i, j) = q(n; i - 1, j) + q(n; i + 1, j) + q(n; i, j - 1) + q(n; i, j + 1) \) (Caloric functions)
Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases

\( q(n; i, j) = \#_{\mathbb{N}^2} \{ (0, 0) \rightarrow^n (i, j) \} \)

\( q(n + 1; i, j) = q(n; i - 1, j) + q(n; i + 1, j) + q(n; i, j - 1) + q(n; i, j + 1) \)

(caloric functions)

\( f(i, j) = \frac{1}{4} \{ f(i - 1, j) + f(i + 1, j) + f(i, j - 1) + f(i, j + 1) \} \)

(preharmonic functions)

Main differences & difficulties

\( \blacksquare \) A unique solution vs. an unknown (\( \leq \infty \)) number of solutions

\( \blacksquare \) Consequence: *guess and prove* techniques do not work
Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases

- \( q(n; i, j) = \#_{\mathbb{N}^2} \{ (0, 0) \xrightarrow{n} (i, j) \} \)
- \( q(n + 1; i, j) = q(n; i - 1, j) + q(n; i + 1, j) + q(n; i, j - 1) + q(n; i, j + 1) \)
  \((\text{Caloric functions})\)
- \( f(i, j) = \frac{1}{4} \{ f(i - 1, j) + f(i + 1, j) + f(i, j - 1) + f(i, j + 1) \} \)
  \((\text{Preharmonic functions})\)

Main differences & difficulties

- A unique solution vs. an unknown \( (\leq \infty) \) number of solutions
- Consequence: \emph{guess and prove} techniques do not work
- Generating functions of preharmonic functions satisfy \emph{kernel functional equations}
- Preharmonic functions \( \approx \) homogenized enumeration problem:

\[
K(x, y)Q(x, y) = K(x, 0)Q(x, 0) + K(0, y)Q(0, y) - K(0, 0)Q(0, 0) - xy
\]

\[
K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0)
\]
Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases

- \( q(n; i, j) = \#_{\mathbb{N}^2} \{(0, 0) \rightarrow^n (i, j)\} \)
- \( q(n + 1; i, j) = q(n; i - 1, j) + q(n; i + 1, j) + q(n; i, j - 1) + q(n; i, j + 1) \)
  \( (\text{Caloric functions}) \)
- \( f(i, j) = \frac{1}{4} \{f(i - 1, j) + f(i + 1, j) + f(i, j - 1) + f(i, j + 1)\} \)
  \( (\text{Preharmonic functions}) \)

Main differences & difficulties

- A unique solution vs. an unknown \( (\leq \infty) \) number of solutions
- Consequence: guess and prove techniques do not work
- Generating functions of preharmonic functions satisfy kernel functional equations
- Preharmonic functions \( \approx \) homogenized enumeration problem:
  \( K(x, y)Q(x, y) = K(x, 0)Q(x, 0) + K(0, y)Q(0, y) - K(0, 0)Q(0, 0) - xy \)
  \( K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \)
- Preharmonic functions \( \rightsquigarrow \) counting numbers asymptotics
Introduction & motivations

Applications in probability theory

Applications in combinatorics

Discrete harmonic functions in the quadrant
Example: construct a 1D process conditioned to stay in \( \mathbb{N} \)
Example: construct a 1D process conditioned to stay in $\mathbb{N}$

Function $f(i) = i$ is positive harmonic and $f(0) = 0$

Weights $p(i, i \pm 1) = \frac{1}{2}$ become $p^f(i, i \pm 1) = \frac{1}{2} \frac{f(i \pm 1)}{f(i)} = \frac{1}{2} \frac{i \pm 1}{i}$
Doob transform 

Example: construct a 1D process conditioned to stay in $\mathbb{N}$

Function $f(i) = i$ is positive harmonic and $f(0) = 0$

Weights $p(i, i \pm 1) = \frac{1}{2}$ become $p^f(i, i \pm 1) = \frac{1}{2} \frac{f(i \pm 1)}{f(i)} = \frac{1}{2} \frac{i \pm 1}{i}$

New weights sum to 1: $f(i - 1) + f(i + 1) = 2f(i)$

Discrete Bessel process

Biane ’90; Mishchenko ’05
**Doob transform**

(1/2)

**Example:** construct a 1D process conditioned to stay in \( \mathbb{N} \)

![Diagram](image)

- Function \( f(i) = i \) is positive harmonic and \( f(0) = 0 \)
- Weights \( p(i, i \pm 1) = \frac{1}{2} \) become \( p^f(i, i \pm 1) = \frac{1}{2} \frac{f(i \pm 1)}{f(i)} = \frac{1}{2} \frac{i \pm 1}{i} \)
- New weights sum to 1: \( f(i - 1) + f(i + 1) = 2f(i) \)

Discrete Bessel process

Biane '90; Mishchenko '05

**Construction can be generalized**

- *Random processes* conditioned never to leave *cones* of \( \mathbb{Z}^d \)
- Quantum random walks, eigenvalues of random matrices, non-colliding random walks, etc.

Dyson '62; Biane '90–'92; Eichelsbacher & König '08
A second way for the conditioning (with the first exit time)

- RW \( \{S(n)\}_{n \geq 0} \); first exit time \( \tau = \inf\{n \geq 0 : S(n) = 0\} \)
A second way for the conditioning (with the first exit time)

- RW \( \{S(n)\}_{n \geq 0} \); first exit time \( \tau = \inf\{n \geq 0 : S(n) = 0\} \)
- On the event \( \{\tau = \infty\} \), the RW stays in \( \{1, 2, 3, \ldots\} \)
A second way for the conditioning (with the first exit time)

- RW \( \{S(n)\}_{n \geq 0} \); first exit time \( \tau = \inf \{ n \geq 0 : S(n) = 0 \} \)
- On the event \( \{ \tau = \infty \} \), the RW stays in \( \{1, 2, 3, ...\} \)
- Replace \( \mathbb{P}[\cdot] \) by \( \mathbb{P}[\cdot | \{\tau = \infty\}] \) to obtain a conditioned RW
A second way for the conditioning (with the first exit time)

- RW \( \{ S(n) \}_{n \geq 0}; \) first exit time \( \tau = \inf\{ n \geq 0 : S(n) = 0 \} \)
- On the event \( \{ \tau = \infty \} \), the RW stays in \( \{1, 2, 3, \ldots\} \)
- Replace \( \mathbb{P}[ \cdot ] \) by \( \mathbb{P}[ \cdot | \{ \tau = \infty \}] \) to obtain a conditioned RW
- Important question: do we have \( \mathbb{P}[ \cdot | \{ \tau = \infty \}] = \mathbb{P}^f \)?
  - \( \text{Denisov & Wachtel '15 (general cones with zero drift); Courtiel, Melczer, Mishna & R. '16 (Gouyou-Beauchamps model with drift)} \)
- Difficulty: exit time \( \tau \)
A second way for the conditioning (with the first exit time)

- RW $\{S(n)\}_{n \geq 0}$; first exit time $\tau = \inf\{n \geq 0 : S(n) = 0\}$
- On the event $\{\tau = \infty\}$, the RW stays in $\{1, 2, 3, ...\}$
- Replace $P[\cdot]$ by $P[\cdot | \{\tau = \infty\}]$ to obtain a conditioned RW
- Important question: do we have $P[\cdot | \{\tau = \infty\}] = P^f$?
  
  - Denisov & Wachtel '15 (general cones with zero drift); Courtiel, Melczer, Mishna & R. '16 (Gouyou-Beauchamps model with drift)

- Difficulty: exit time $\tau$

Example (1/3) in the quadrant: the simple walk

- Uniform weights $\frac{1}{4}$
- $f(i, j) = i \cdot j$
- Unique preharmonic function (up to multiplicative factors)
- Product form  
  
  - Picardello & Woess '92
A second way for the conditioning (with the first exit time)

- RW $\{S(n)\}_{n \geq 0}$; first exit time $\tau = \inf\{n \geq 0 : S(n) = 0\}$
- On the event $\{\tau = \infty\}$, the RW stays in $\{1, 2, 3, \ldots\}$
- Replace $\mathbb{P}[\cdot]$ by $\mathbb{P}[\cdot | \{\tau = \infty\}]$ to obtain a conditioned RW
- Important question: do we have $\mathbb{P}[\cdot | \{\tau = \infty\}] = \mathbb{P}^f$?
  - Denisov & Wachtel '15 (general cones with zero drift); Courtiel, Melczer, Mishna & R. '16 (Gouyou-Beauchamps model with drift)
- Difficulty: exit time $\tau$

Example (2/3) in the quadrant: the Tandem walk

- Uniform weights $\frac{1}{3}$
- $f(i, j) = i \cdot j \cdot (i + j)$
- Unique preharmonic function (up to multiplicative factors)
  - Biane '92
A second way for the conditioning (with the first exit time)

- RW \(\{S(n)\}_{n \geq 0}\); first exit time \(\tau = \inf\{n \geq 0 : S(n) = 0\}\)
- On the event \(\{\tau = \infty\}\), the RW stays in \(\{1, 2, 3, \ldots\}\)
- Replace \(\mathbb{P}[\cdot]\) by \(\mathbb{P}[\cdot | \{\tau = \infty\}]\) to obtain a conditioned RW
- Important question: do we have \(\mathbb{P}[\cdot | \{\tau = \infty\}] = \mathbb{P}^f\)?
  - Denisov & Wachtel '15 (general cones with zero drift); Courtiel, Melczer, Mishna & R. '16 (Gouyou-Beauchamps model with drift)
- Difficulty: exit time \(\tau\)

Example (3/3) in the quadrant: the GB walk

- Uniform weights \(\frac{1}{4}\)
- \(f(i,j) = i \cdot j \cdot (i + j) \cdot (i + 2j)\)
- Unique preharmonic function (up to multiplicative factors)
  - Biane '92
Martin boundary theory

Rough description

- *Martin boundary* $\partial \rightsquigarrow$ set of all harmonic functions

---

Minimal Martin boundary $\partial_m$; integral representation of all harmonic functions:

$$f = \int_{\partial_m} \{ \text{Martin kernel} \} \, d\mu_f$$

Martin ’41; Hunt ’57; Doob ’59; Choquet & Deny ’60; Ney & Spitzer ’66; Picardello & Woess ’92

Homogeneous random processes

- Well understood
  - Spitzer ’64; Ney & Spitzer ’66

Non-homogeneous random processes:

- Difficult problem
  - Walks related to Lie algebras
    - Biane ’90–’92
  - Quadrant walks with drift
    - Ignatiouk-Robert ’10
  - Quadrant walks with zero drift
    - Partial results by R. ’14; Bouaziz, Mustapha & Sifi ’15
  - General RW in cones: open problem (conjecture: uniqueness $\iff$ drift = 0)
Martin boundary theory

Rough description

- **Martin boundary** $\partial \sim$ set of all harmonic functions
- **Minimal Martin boundary** $\partial_m \sim$ integral representation of all harmonic functions: $f = \int_{\partial_m} \{\text{Martin kernel}\} d\mu_f$

References:
- Martin '41;
- Hunt '57; Doob '59; Choquet & Deny '60; Ney & Spitzer '66; Picardello & Woess '92

- Homogeneous random processes
  - Well understood
    - Spitzer '64; Ney & Spitzer '66

- Non-homogeneous random processes:
  - Difficult problem
    - Walks related to Lie algebras
      - Biane '90–'92
    - Quadrant walks with drift
      - Ignatiouk-Robert '10
    - Quadrant walks with zero drift
      - Partial results by R. '14; Bouaziz, Mustapha & Sifi '15
    - General RW in cones: open problem (conjecture: uniqueness $\iff$ drift = 0)
Martin boundary theory

Rough description

- **Martin boundary** $\partial \sim$ set of all harmonic functions
- **Minimal Martin boundary** $\partial_m \sim$ integral representation of all harmonic functions: $f = \int_{\partial_m} \{\text{Martin kernel}\} d\mu_f$

Homogeneous random processes

- Well understood

Non-homogeneous random processes:
  - Walks related to Lie algebras
    - Biane '90–'92
  - Quadrant walks with drift
    - Ignatiouk-Robert '10
  - Quadrant walks with zero drift
    - Partial results by R. '14; Bouaziz, Mustapha & Sifi '15
  - General RW in cones: open problem (conjecture: uniqueness $\iff$ drift = 0)

References:

- Martin '41; Hunt '57; Doob '59; Choquet & Deny '60; Ney & Spitzer '66; Picardello & Woess '92
- Spitzer '64; Ney & Spitzer '66
Martin boundary theory

Rough description

- Martin boundary $\partial \sim$ set of all harmonic functions
- Minimal Martin boundary $\partial_m \sim$ integral representation of all harmonic functions: $f = \int_{\partial_m} \left\{\text{Martin kernel}\right\} d\mu_f$

Martin '41; Hunt '57; Doob '59; Choquet & Deny '60; Ney & Spitzer '66; Picardello & Woess '92

Homogeneous random processes

- Well understood

Spitzer '64; Ney & Spitzer '66

Non-homogeneous random processes: difficult problem

- Walks related to Lie algebras
  
- Quadrant walks with drift $\sum_{y \in \mathbb{N}} y \cdot p(y)$
  
- Quadrant walks with zero drift
    
    Partial results by R. '14; Bouaziz, Mustapha & Sifi '15

Biane '90–'92

Ignatiouk-Robert '10
Martin boundary theory

**Rough description**

- **Martin boundary** $\partial \sim$ set of all harmonic functions
- **Minimal Martin boundary** $\partial_m \sim$ integral representation of all harmonic functions: $f = \int_{\partial_m} \{\text{Martin kernel}\} d\mu_f$

**Homogeneous random processes**

- Well understood

**Non-homogeneous random processes:** difficult problem

- Walks related to Lie algebras
- Quadrant walks with drift $\sum_{y \in \mathbb{N}} y \cdot p(y)$
- Quadrant walks with zero drift
  Partial results by R. ’14; Bouaziz, Mustapha & Sifi ’15
- General RW in cones: open problem (conjecture: uniqueness $\iff$ drift $= 0$)
Introduction & motivations

Applications in probability theory

Applications in combinatorics

Discrete harmonic functions in the quadrant
Asymptotics of some numbers of walks

Asymptotic statements

- **Total number of walks** starting at \((k, \ell)\):

  \[
  q(n; k, \ell; \mathbb{N}^2) = \#_{\mathbb{N}^2} \{(k, \ell) \xrightarrow{n} \mathbb{N}^2\} \\
  \sim f_1(k, \ell) \cdot \rho_1^n \cdot n^{\alpha_1}
  \]

  🛠 Not proved yet!
Asymptotics of some numbers of walks

Asymptotic statements

- **Total number of walks** starting at \((k, \ell)\):
  \[
  q(n; k, \ell; \mathbb{N}^2) = \#_{\mathbb{N}^2}\{(k, \ell) \xrightarrow{n} \mathbb{N}^2\}
  \sim f_1(k, \ell) \cdot \rho_1^n \cdot n^{\alpha_1}
  
  \]
  Not proved yet!

- **Excursions** starting at \((k, \ell)\):
  \[
  q(n; k, \ell; i, j) = \#_{\mathbb{N}^2}\{(k, \ell) \xrightarrow{n} (i, j)\}
  \sim f_2(k, \ell) \cdot f_2'(i, j) \cdot \rho_2^n \cdot n^{\alpha_2}
  
  \]
  Denisov & Wachtel '15

Application: random maps with bipolar orientations
Asymptotics of some numbers of walks

Asymptotic statements

- **Total number of walks** starting at \((k, \ell)\):
  \[
  q(n; k, \ell; \mathbb{N}^2) = \#_{\mathbb{N}^2}\{(k, \ell) \xrightarrow{n} \mathbb{N}^2\} \\
  \sim f_1(k, \ell) \cdot \rho_1^n \cdot n^{\alpha_1}
  \]
  *Not proved yet!*

- **Excursions** starting at \((k, \ell)\):
  \[
  q(n; k, \ell; i, j) = \#_{\mathbb{N}^2}\{(k, \ell) \xrightarrow{n} (i, j)\} \\
  \sim f_2(k, \ell) \cdot f_2'(i, j) \cdot \rho_2^n \cdot n^{\alpha_2}
  \]
  *Denisov & Wachtel ’15*

**Application:** random maps with bipolar orientations

*© J. Bettinelli*

*Bousquet-Mélou, Fusy & R. ’17*
Random generation

**Aim:** generate efficiently a long walk (e.g., confined to a region)

A walk of length 18000

---

Different methods

- **Recursive method** (step-by-step construction)
- **Bijections** (if existing) For Kreweras see Bernardi '07
- **Rejection algorithms** Bacher & Sportiello '16; Lumbroso, Mishna & Ponty '16
- **Preharmonic functions and Doob transform** Fusy '16

(Difficulty: after Doob transform, non-uniform walks)
Random generation

**Aim:** generate efficiently a long walk (e.g., confined to a region)

A walk of length 18000

**Different methods**

- **Recursive method** (step-by-step construction)
- **Bijections** (if existing) For Kreweras see 📘 Bernardi '07
- **Rejection algorithms** 📘 Bacher & Sportiello '16; Lumbroso, Mishna & Ponty '16
Random generation

**Aim:** generate efficiently a long walk (e.g., confined to a region)

A walk of length 18000

**Different methods**

- *Recursive method* (step-by-step construction)
- *Bijections* (if existing) For Kreweras see \(\textcircled{\text{ Bernardi '07}}\)
- *Rejection algorithms* \(\textcircled{\text{ Bacher & Sportiello '16; Lumbroso, Mishna & Ponty '16}}\)
- Preharmonic functions and *Doob transform* \(\textcircled{\text{ Fusy '16}}\)
  (Difficulty: after Doob transform, non-uniform walks)
Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  D’Arco, Lacivita & Mustapha ’16
- Asymptotics in three quarters of plane  
  Mustapha ’16

Zero drift case: classical inequalities

Varopoulos ’99–’09

General principle: there is a canonical function (the réduite of the cone $f$: $\Delta [f] = 0$) containing “all” the information

$\sum_{n; k, \ell; N} \approx f(k, \ell) \cdot \rho^n \cdot n^{\alpha}$ as $n \to \infty$

$\alpha$ = homogeneity degree of $f$

$\sim f$ asymptotically

Non-zero drift case: Cramér’s transform & ongoing work

Works if drift with $\leq 0$ coordinates

Ongoing work in the remaining cases

Garbit, Mustapha & R.
Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  D’Arco, Lacivita & Mustapha '16
- Asymptotics in three quarters of plane  
  Mustapha '16

Zero drift case: classical inequalities  
Varopoulos '99–'09

General principle: there is a canonical function (the réduite of the cone $f_c$: $\Delta[f_c] = 0$) containing “all” the information
Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  D’Arco, Lacivita & Mustapha ’16
- Asymptotics in three quarters of plane  
  Mustapha ’16

Zero drift case: classical inequalities  
Varopoulos ’99–’09

General principle: there is a canonical function (the réduite of the cone $f_c$: $\Delta[f_c] = 0$) containing “all” the information:

- $q(n; k, \ell; \mathbb{N}^2) \approx f(k, \ell) \cdot \rho^n \cdot n^\alpha$ as $n \to \infty$
Potential theoretic tools

**Counting numbers are caloric functions**

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  D’Arco, Lacivita & Mustapha '16
- Asymptotics in three quarters of plane  
  Mustapha '16

**Zero drift case:** classical inequalities  
Varopoulos ’99–’09

General principle: there is a canonical function (the réduit of the cone \( f_c \): \( \Delta[f_c] = 0 \)) containing “all” the information:

- \( q(n; k, \ell; \mathbb{N}^2) \approx f(k, \ell) \cdot \rho^n \cdot n^\alpha \) as \( n \to \infty \)
- \( \alpha = \) homogeneity degree of \( f_c \)
Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  D’Arco, Lacivita & Mustapha ’16
- Asymptotics in three quarters of plane  
  Mustapha ’16

Zero drift case: classical inequalities  
Varopoulos ’99–’09

General principle: there is a canonical function (the réduite of the cone $f_c$: $\Delta[f_c] = 0$) containing “all” the information:

- $q(n; k, \ell; \mathbb{N}^2) \approx f(k, \ell) \cdot \rho^n \cdot n^\alpha$ as $n \to \infty$
- $\alpha = \text{homogeneity degree of } f_c$
- $f \sim f_c$ asymptotically
Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities)  
  
  D’Arco, Lacivita & Mustapha ’16

- Asymptotics in three quarters of plane  
  
  Mustapha ’16

Zero drift case: classical inequalities  

Varopoulos ’99–’09

General principle: there is a canonical function (the réduite of the cone $f_c$: $\Delta[f_c] = 0$) containing “all” the information:

- $q(n; k, \ell; \mathbb{N}^2) \approx f(k, \ell) \cdot \rho^n \cdot n^\alpha$ as $n \to \infty$

- $\alpha = \text{homogeneity degree of } f_c$

- $f \sim f_c$ asymptotically

Non-zero drift case: Cramér’s transform & ongoing work

- Works if drift with $\leq 0$ coordinates

- Ongoing work in the remaining cases  
  
  Garbit, Mustapha & R.
Introduction & motivations

Applications in probability theory

Applications in combinatorics

Discrete harmonic functions in the quadrant
**Functional equation & Tutte’s invariants**

**A functional equation** reminiscent of the enumeration

- \( F(x, y) = \sum_{i,j \geq 1} f(i,j)x^{i-1}y^{j-1} \)
- \( K'(x, y) = xy\{\sum_{-1 \leq k, \ell \leq 1} p(k, \ell)x^{-k}y^{-\ell} - 1\} \)

**Kernel functional equation:**

\[
K'(x, y)F(x, y) =
K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0)
\]
Functional equation & Tutte’s invariants

A functional equation reminiscent of the enumeration

- $F(x, y) = \sum_{i,j \geq 1} f(i, j)x^{i-1}y^{j-1}$
- $K'(x, y) = xy\{\sum_{-1 \leq k, \ell \leq 1} p(k, \ell)x^{-k}y^{-\ell} - 1\}$
- **Kernel functional equation:**
  \[
  K'(x, y)F(x, y) = \quad K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0)
  \]

Definition of Tutte’s invariants

- Introduced to count $q$-colored triangulations & planar maps
  \(\text{Tutte ’73; Bernardi & Bousquet-Mélou ’11}\)
- Define $X_0$ & $X_1$ by $K'(X_0, y) = K'(X_1, y) = 0$
- Tutte’s invariant: function $I \in \mathbb{Q}[\![x]\!]$ such that $I(X_0) = I(X_1)$
A functional equation reminiscent of the enumeration

\[ F(x, y) = \sum_{i,j \geq 1} f(i, j)x^{i-1}y^{j-1} \]
\[ K'(x, y) = xy\{\sum_{-1 \leq k, \ell \leq 1} p(k, \ell)x^{-k}y^{-\ell} - 1\} \]
\[ \text{Kernel functional equation:} \]
\[ K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \]

Definition of Tutte’s invariants

- Introduced to count \( q \)-colored triangulations & planar maps
  - Tutte ’73; Bernardi & Bousquet-Mélou ’11
- Define \( X_0 \) & \( X_1 \) by \( K'(X_0, y) = K'(X_1, y) = 0 \)
- Tutte’s invariant: function \( I \in \mathbb{Q}[[x]] \) such that \( I(X_0) = I(X_1) \)

The sections \( K'(x, 0)F(x, 0) \) & \( K'(0, y)F(0, y) \) are invariants

- Evaluate the functional equation at \( X_0 \) & \( X_1 \)
- Make the difference of the two identities
**Functional equation & Tutte’s invariants**

**A functional equation** reminiscent of the enumeration

- $F(x, y) = \sum_{i,j \geq 1} f(i,j)x^{i-1}y^{j-1}$
- $K'(x, y) = xy\{\sum_{-1 \leq k, \ell \leq 1} p(k, \ell)x^{-k}y^{-\ell} - 1\}$
- **Kernel functional equation:**
  
  \[
  K'(x, y)F(x, y) = \\
  K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0)
  \]

**Definition of Tutte’s invariants**

- Introduced to count $q$-colored triangulations & planar maps
  
  \(\text{Tutte '73; Bernardi & Bousquet-Mélo '11}\)
- Define $X_0$ & $X_1$ by $K'(X_0, y) = K'(X_1, y) = 0$
- Tutte’s invariant: function $I \in \mathbb{Q}[[x]]$ such that $I(X_0) = I(X_1)$

**The sections** $K'(x, 0)F(x, 0)$ & $K'(0, y)F(0, y)$ are invariants

- Evaluate the functional equation at $X_0$ & $X_1$
- Make the difference of the two identities

**Does this characterize the sections?**
Tutte’s invariants & conformal mappings

A general theorem: uniqueness of positive harmonic functions

\[ K'(x, 0)F(x, 0) = w(x), \text{ characterized by} \]

- Conformal mapping of a certain domain
- \( w(x) = w(\bar{x}) \)
- \( w(1) = \infty \)
- Same for \( K'(0, y)F(0, y) \)
**Tutte’s invariants & conformal mappings**

**A general theorem:** description of all harmonic functions

\[ K'(x, 0)F(x, 0) = \text{Poly}(w(x)), \text{ charac. by} \]

- Conformal mapping of a certain domain
- \( w(x) = w(\bar{x}) \)
- \( w(1) = \infty \)
- Same for \( K'(0, y)F(0, y) \)

**Question:** How deep is this connection conformal maps/harmonic functions?
Tutte’s invariants & conformal mappings

A general theorem: description of all harmonic functions

\[ K'(x, 0)F(x, 0) = \text{Poly}(w(x)), \text{ charac. by} \]

- Conformal mapping of a certain domain
- \( w(x) = w(\overline{x}) \)
- \( w(1) = \infty \)
- Same for \( K'(0, y)F(0, y) \)

Going back to the SRW

\[ K'(x, 0)F(x, 0) = \frac{x}{4(1-x)^2}, \text{ characterized by} \]

- Conformal mapping of the unit disc
- \( w(e^{i\theta}) = w(e^{-i\theta}) \)
- \( w(1) = \infty \)
- Same for \( K'(0, y)F(0, y) = \frac{y}{4(1-y)^2} \)
Tutte’s invariants & conformal mappings

A general theorem: description of all harmonic functions

\[ K'(x, 0) F(x, 0) = \text{Poly}(w(x)), \] charac. by

- Conformal mapping of a certain domain
  - \( w(x) = w(\overline{x}) \)
  - \( w(1) = \infty \)
  - Same for \( K'(0, y) F(0, y) \)

Going back to the SRW

\[ K'(x, 0) F(x, 0) = \frac{x}{4(1-x)^2}, \] characterized by

- Conformal mapping of the unit disc
  - \( w(e^{i\theta}) = w(e^{-i\theta}) \)
  - \( w(1) = \infty \)
  - Same for \( K'(0, y) F(0, y) = \frac{y}{4(1-y)^2} \)

Question

How deep is this connection conformal maps/harmonic functions?
THANK YOU
Example: the SRW

A product-form generating function

\[ f(i, j) = i \cdot j \implies F(x, y) = \sum_{i, j \geq 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2} \]

Kernel: \[ K'(x, y) = xy \left\{ \frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1 \right\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4} \]
Example: the SRW

A product-form generating function

\[ f(i, j) = i \cdot j \implies F(x, y) = \sum_{i,j \geq 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2} \]

Kernel: \( K'(x, y) = xy\left\{\frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1\right\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4} \)

Verification of the functional equation

\[ K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \]
Example: the SRW

A product-form generating function

\[ f(i, j) = i \cdot j \implies F(x, y) = \sum_{i,j \geq 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2} \]

Kernel: \( K'(x, y) = xy \left\{ \frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1 \right\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4} \)

Verification of the functional equation

\[ K'(x, y)F(x, y) - K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \]
\[ = \frac{x}{4} \times \frac{1}{(1-x)^2} + \frac{y}{4} \times \frac{1}{(1-y)^2} - 0 \times 1 \]
Example: the SRW

A product-form generating function

\[ f(i, j) = i \cdot j \implies F(x, y) = \sum_{i, j \geq 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2} \]

Kernel: \( K'(x, y) = xy \left\{ \frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1 \right\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4} \)

Verification of the functional equation

\[ K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \]
\[ = \frac{x}{4} \times \frac{1}{(1-x)^2} + \frac{y}{4} \times \frac{1}{(1-y)^2} - 0 \times 1 \]

Tutte’s invariants

\[ I(X_0) = I(X_1) \xrightarrow{X_0X_1=1} I(x) = I\left(\frac{1}{x}\right) \implies I \text{ function of } x + \frac{1}{x} \]
\[ K'(x, 0)F(x, 0) = \frac{x}{4} \frac{1}{(1-x)^2} = \frac{1}{4} \frac{1}{x+\frac{1}{x}-2} \text{ is an invariant} \]
Example: the SRW

A product-form generating function

\[ f(i, j) = i \cdot j \implies F(x, y) = \sum_{i,j\geq 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2} \]

Kernel: \( K'(x, y) = xy\left\{ \frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1 \right\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4} \)

Verification of the functional equation

\[ K'(x, y)F(x, y) = K'(x, 0)F(x, 0) + K'(0, y)F(0, y) - K'(0, 0)F(0, 0) \]
\[ = \frac{x}{4} \times \frac{1}{(1-x)^2} + \frac{y}{4} \times \frac{1}{(1-y)^2} - 0 \times 1 \]

Tutte's invariants

\[ I(X_0) = I(X_1) \xrightarrow{X_0X_1=1} I(x) = I\left(\frac{1}{x}\right) \implies I \text{ function of } x + \frac{1}{x} \]
\[ K'(x, 0)F(x, 0) = \frac{x}{4} \times \frac{1}{(1-x)^2} = \frac{1}{4} \times \frac{1}{x+\frac{1}{x}-2} \text{ is an invariant} \]

Why this function of \( x + \frac{1}{x} \)?

- Of order 1 in \( x + \frac{1}{x} \) \( \sim \) Minimality (conformal mappings)
- \( F(1, 0) = \infty \sim \) Liouville's theorem